

Some remarks on generalized roundness

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Abstract By using the links between generalized roundness, negative type inequalities and equivariant Hilbert space compressions, we obtain that the generalized roundness of the usual Cayley graph of finitely generated free groups and free abelian groups of rank ≥ 2 equals 1. This answers a question of J-F. Lafont and S. Prassidis.

Keywords Generalized roundness · Negative type functions · Hilbert space compression · CAT(0) cube complexes

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1 Introduction

Generalized roundness (see definition below) was introduced by Enflo in [5] and [6] in order to study the uniform structure of metric spaces, and as an application of this notion he gave a solution to Smirnov's problem [6]. Rudiments of a general theory for generalized roundness were developed in [9], where the link of this notion with negative type inequalities is emphasized. More recently, generalized roundness was investigated in the case of finitely generated groups [10]. Unfortunately, generalized roundness is very difficult to estimate in general, and for this reason there are only very few examples of metric spaces for which the exact value is known. Here we use ideas developed in [9] together with estimates on generalized roundness computed in [10] and results about equivariant Hilbert space compression in [8] to deduce exact values of the generalized roundness of finitely generated free (abelian and non-abelian) groups endowed with their standard metrics.

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2 Preliminaries

Let (X, d) be a metric space, and let G denote a group acting on X by isometries.

Definition 2.1 The *generalized roundness* of (X, d) is the supremum of all positive numbers p such that for every $n \geq 2$ and any collection of $2n$ points $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ in X , the following inequality holds:

$$\sum_{1 \leq i < j \leq n} (d(a_i, a_j)^p + d(b_i, b_j)^p) \leq \sum_{1 \leq i, j \leq n} d(a_i, b_j)^p.$$

We will denote the generalized roundness of the metric space (X, d) by $\text{gr}(X, d)$, and simply $\text{gr}(X)$ when there is no ambiguity about the metric d .

Essentially, a metric space (X, d) satisfies $\text{gr}(X, d) = p$ if $2n$ -gons (for every $n \geq 2$) are thinner than the ones in L^p -spaces. This observation is justified by the following result (see [9]):

Proposition 2.2 Let $0 < p \leq 2$ and (X, \mathcal{B}, μ) be a measured space. Then $\text{gr}(L^p(X, \mathcal{B}, \mu)) = p$.

Remark 2.3 The generalized roundness of any infinite and finitely generated group (endowed with the word metric) is always ≤ 2 (see [10] Proposition 4.7).

Definition 2.4 A function $\psi: X \times X \rightarrow \mathbb{R}$ is said to be a *kernel of negative type* if $\psi(x, x) = 0$ for all $x \in X$, $\psi(x, y) = \psi(y, x)$ for all $x, y \in X$, and if for every integer $n \geq 1$, for every $x_1, \dots, x_n \in X$ and for every $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\sum_{i=1}^n \lambda_i = 0$, the following inequality holds:

$$\sum_{1 \leq i, j \leq n} \lambda_i \lambda_j \psi(x_i, x_j) \leq 0.$$

The kernel is said to be G -invariant if $\psi(gx, gy) = \psi(x, y)$ for all $x, y \in X$ and for all $g \in G$.

Kernels of negative type and generalized roundness are related by the following result (see [9]):

Theorem 2.5 $\text{gr}(X, d) \geq p$ if and only if d^p is a kernel of negative type.

Definition 2.6 Let \mathcal{H} be an Hilbert space. A map $f: X \rightarrow \mathcal{H}$ is said to be a *uniform embedding* of X into \mathcal{H} if there exist non-decreasing functions $\rho_{\pm}(f): \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

- (i) $\rho_{-}(f)(d(x, y)) \leq \|f(x) - f(y)\|_{\mathcal{H}} \leq \rho_{+}(f)(d(x, y))$, for all $x, y \in X$;
- (ii) $\lim_{r \rightarrow +\infty} \rho_{\pm}(f)(r) = +\infty$.

Then the G -equivariant Hilbert space compression of the metric space X , denoted by $R_G(X)$, is defined as the supremum of all $0 < \beta \leq 1$ for which there exists a G -equivariant uniform embedding f into some Hilbert space which is equipped with an action of G by affine isometries, such that $\rho_{+}(f)$ is affine and $\rho_{-}(f)(r) = r^{\beta}$ (for large enough r).

Concerning negative definite kernels, we will need a G -invariant analogue of the so-called GNS-construction (see for instance [2], 2.10):

Proposition 2.7 *Let ψ be a G -invariant kernel of negative type on X , then there exists a Hilbert space \mathcal{H} equipped with an action of G by affine isometries, and a G -equivariant map $f: X \rightarrow \mathcal{H}$, such that $\psi(x, y) = \|f(x) - f(y)\|_{\mathcal{H}}^2$ for all $x, y \in X$.*

Theorem 2.5 combined with Proposition 2.7 immediately gives the following estimate:

Proposition 2.8 $R_G(X) \geq \frac{\text{gr}(X)}{2}$.

Remark 2.9 On one hand, the previous inequality cannot be improved. Indeed, let $X = G = \mathbb{Z}$ acting on itself by left translations and being endowed with its usual left invariant word metric. Considering the inclusion of \mathbb{Z} into \mathbb{R} , which is a \mathbb{Z} -equivariant isometry, we have $R_{\mathbb{Z}}(\mathbb{Z}) = 1$ and moreover $\text{gr}(\mathbb{Z}) \geq \text{gr}(\mathbb{R})$. But Proposition 2.2 gives $\text{gr}(\mathbb{R}) = 2$. Therefore, by Remark 2.3, we obtain that $\text{gr}(\mathbb{Z}) = 2 = 2R_{\mathbb{Z}}(\mathbb{Z})$.

On the other hand, the inequality is unfortunately not an equality in general. Consider for instance the case $X = G = \mathbb{Z}^2$. The Hilbert space compression of \mathbb{Z}^2 equals 1 (see [8] Example 2.7), and by amenability the equivariant Hilbert space compression of \mathbb{Z}^2 equals the Hilbert space compression (see [4] Proposition 4.4). Hence $R_{\mathbb{Z}^2}(\mathbb{Z}^2) = 1$. But, by Corollary 3.2 below, $\text{gr}(\mathbb{Z}^2) = 1$.

3 Negative type inequalities in CAT(0) cube complexes

Recall that a cube complex is a metric polyhedral complex in which each cell is isometric to an Euclidean cube $[-\frac{1}{2}, \frac{1}{2}]^n$, and the gluing maps are isometries. A finite dimensional cube complex always carries a complete geodesic metric (see [1]). A cube complex is CAT(0) if it is simply connected and if, in the link of every cube of the complex, there is at most one edge between any two vertices and there is no triangle not contained in a 2-simplex (see [1] and [7]). Let X denote a finite dimensional CAT(0) cube complex. The 0-skeleton $X^{(0)}$ of X can be endowed with the metric, denoted by d_0 , given by the length of the shortest edge path in the 1-skeleton of X between vertices. The proof of the next result is strongly inspired by [3] Example 1.

Theorem 3.1 *Let X be a finite dimensional CAT(0) cube complex. Then $\text{gr}(X^{(0)}, d_0) \geq 1$.*

Proof By Proposition 2.2, it is sufficient to exhibit an isometric embedding of $(X^{(0)}, d_0)$ into some L^1 -space. Given an edge in the complex, there is a unique isometrically embedded codimension 1 coordinate hyperplane (again called hyperplane) which cuts this edge transversely in its midpoint, and this hyperplane separates the complex into two components, called half spaces (see [7]). We will denote by H the set of all hyperplanes. Moreover, by [11], shortest edge paths in the 1-skeleton cross any hyperplane at most once. Hence the distance between two vertices, $d_0(v, w)$, is the number of hyperplanes separating v and w (hyperplanes such that the two vertices are not in the same half space). We fix a vertex $v_0 \in X^{(0)}$ and for every vertex $v \in X^{(0)}$ we set $H_v := \{h \in H \mid h \text{ separates } v_0 \text{ and } v\}$. Then we define

$$f: X^{(0)} \rightarrow l^1(H), \quad v \mapsto \sum_{h \in H_v} \delta_h$$

where

$$\delta_h: H \rightarrow \mathbb{R}, \quad k \mapsto \begin{cases} 1 & \text{if } k = h \\ 0 & \text{otherwise} \end{cases}$$

It remains to show that f is an isometry. Let v, w be two vertices of X . Then

$$\|f(v) - f(w)\|_{l^1(H)} = \sum_{l \in H} \left| \sum_{h \in H_v} \delta_h(l) - \sum_{h \in H_w} \delta_h(l) \right|. \quad (*)$$

For every $l \in H$, we have

$$\left| \sum_{h \in H_v} \delta_h(l) - \sum_{h \in H_w} \delta_h(l) \right| = \begin{cases} 1 & \text{if } l \in H_v \triangle H_w \\ 0 & \text{if } l \in H_v \cap H_w \end{cases}$$

But a hyperplane $l \in H$ separates v and w if and only if $l \in H_v \triangle H_w$. Hence, the sum in the left member of $(*)$ is exactly the number of hyperplanes separating v and w , i.e., $\|f(v) - f(w)\|_{l^1(H)} = d_0(v, w)$. \square

Corollary 3.2 *Let $n \geq 2$. We endow \mathbb{Z}^n with the word metric associated to its canonical basis, and we endow the free group of rank n , F_n , with the word metric associated to any free generating system. We have:*

- (i) $\text{gr}(\mathbb{Z}^n) = 1$;
- (ii) $\text{gr}(F_n) = 1$.

Proof (i) By Corollary 4.14 of [10], we have $\text{gr}(\mathbb{Z}^n) \leq 1$. For the converse inequality, let us consider the action of \mathbb{Z}^n on \mathbb{R}^n by left translations. \mathbb{R}^n can be viewed naturally as a CAT(0) cube complex X of which the 0-skeleton (endowed with the metric d_0) is isometric to \mathbb{Z}^n . Therefore, Theorem 3.1 gives the result.

(ii) The Cayley graph of F_n is a tree. In particular, this is a 1-dimensional CAT(0) cube complex. Hence, by Theorem 3.1, we obtain that $\text{gr}(F_n) \geq 1$. On the other hand, it is known that $R_{F_n}(F_n) = \frac{1}{2}$ (see [8]). Then by Proposition 2.8, we deduce that $\text{gr}(F_n) \leq 1$. \square

Remark 3.3 Let G be a group acting freely by isometries on the 0-skeleton $(X^{(0)}, d_0)$ of a CAT(0) cube complex X . If we fix a vertex v_0 , we define a metric D_0 on G by setting $D_0(g, h) := d_0(gv_0, hv_0)$, and Theorem 3.1 gives $\text{gr}(G, D_0) \geq 1$.

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